MATH 2020 A
Fall, 2020.

Lecture 18

4 Applications of Green's theorem

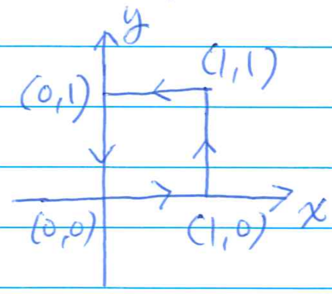
- Use double $\int$ to evaluate line integrals
- area formula
- compatibility conditions
- divergence and rotation

(1) Green's formula relates double integrals to line integrals.

e.g. Evaluate

$$\oint_C -y^2\,dx + xy\,dy$$

where C is the boundary of the square at $(0,0), (1,0), (1,1), (0,1)$ in positive direction (anticlockwise).



Need to do four line integrals.
So better use the other side, i.e., convert to double integral.

$P = -y^2$, $Q = xy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - (-2y) = 3y.$$

$$\therefore \oint_C -y^2\,dx + xy\,dy = \iint_R 3y\,dA(x,y) = \int_0^1\int_0^1 3y\,dx\,dy$$

$$= \int_0^1 3y\,dy = 3/2 \ \#$$

(2) Area formula. Take $P = y$ and $Q = 0$ in the Green's formula:

$$\iint_D -1 \, dA = \oint_C y\,dx$$

$$\therefore A = -\oint_C y\,dx$$ (A is the area enclosed by C)

Take $P=0$ and $Q=x$,

$$\iint_D 1 \, dA = \oint x\,dy$$

$$\therefore A = \oint_C x\,dy.$$

Putting together to get a more symmetric form,

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy \qquad (*).$$

This formula express area as a line integral along the boundary of the region, It has theoretical interest. For instance, one can use it to prove the isoperimetric inequality:

$4\pi A \le L^2$ "=" holds iff C is a circle. Here

L is the perimeter of C.

(3) In last lectures, we showed

(a) $\vec{F} = (P, Q)$ is conservative iff $\vec{F}$ is indep't of path

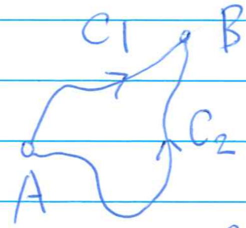
(b) $\vec{F}$ is conservative $\Longrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Now, we can prove

Theorem 3 $(n=2)$ Suppose G is simply connected. Then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \text{ is conservative.}$$

Pf. Let C_1 and C_2 be 2 paths from A to B in G. Need to check



$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$$

Suppose first that $C_1 \cap C_2 = \phi$ (except at A, B)

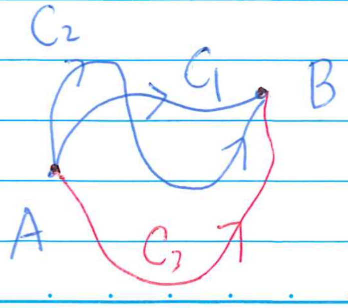
then $C = C_2 - C_1$ is a simple closed loop whose enclosed set D completely contained in G ($\because$ G has no holes). By Green's

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\, dA = 0.$$

$$\therefore \oint_{C_2} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy.$$

Now, C_1 and C_2 may intersect. Then we choose C_3 from A to B such that $C_1 \cap C_3 = \phi$, $C_1 \cap C_2 = \phi$ (except at A, B). Then

$$\oint_{C_1} \cdots = \oint_{C_3}, \quad \oint_{C_2} \cdots = \oint_{C_3} \Rightarrow \oint_{C_1} (\cdots) = \oint_{C_2} (\ ). \quad \#$$



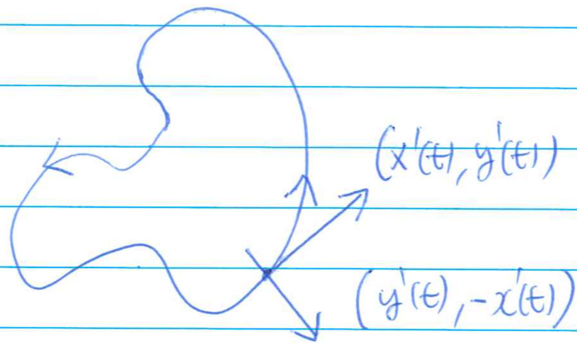
⇒) done already. Recall, if $F = \nabla\Phi$, then $P = \frac{\partial \Phi}{\partial x}$, $Q = \frac{\partial \Phi}{\partial y}$

so $\frac{\partial P}{\partial y} = \frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial Q}{\partial x}$.

Here we don't need G to be simply-connected.

(4) Recall that for a simple closed loop C in positive direction, the circulation of a v.f. $\vec{F}$ on C is

$$\oint_C P\,dx + Q\,dy .$$



the flux of $\vec{F}$ (w.r.t. outward normal) is

$$\oint P\,dy - Q\,dx$$

unit outer normal $\hat{n}$ $\frac{(y', -x')}{\sqrt{x'^2+y'^2}}$

circulation : $(P, Q)\cdot(x', y')$

flux : $(P, Q)\cdot(y', -x')$

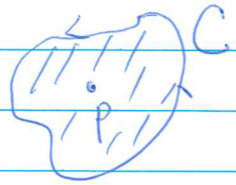
Green's thm suggests a way to define the circulation and the flux of a v.f. at a single point.

Let $p = (x, y)$ be a point in open set G where a C^1-v.f. $\vec{F} = (P, Q)$ is defined. Let C be a simple closed loop enclosing the point p. Then the circulation of $\vec{F}$ around C is

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where D is the enclosed set of C, and $p \in D$

Divide both side by $|D|$, the area of D,



$$\frac{1}{|D|}\oint_C P\,dx + Q\,dy = \frac{1}{|D|}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\to \frac{\partial Q}{\partial x}(p) - \frac{\partial P}{\partial y}(p)$$ as C shrinks to the pt p.

It suggests to define the rotation (or the curl) of $\vec{F}$ at p to be

$$\operatorname{rot} \vec{F}(p) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(p).$$

Next, the flux of $\vec{F}$ through C is

$$\oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

So $$\frac{1}{|D|}\oint_C P\,dy - Q\,dx = \frac{1}{|D|}\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$\to \frac{\partial P}{\partial x}(p) + \frac{\partial Q}{\partial y}(p) \text{ as } C \text{ shrinks to } p.$$

It suggest to define the divergence of $\vec{F}$ at p to be

$$\operatorname{div} \vec{F}(p) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)(p).$$

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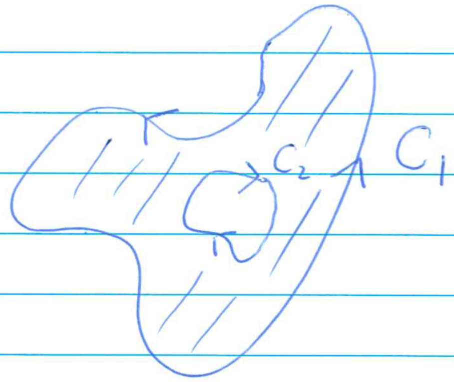
A more general Green's Theorem.
Let's consider a simple case where D is bdd between 2 closed curves.



C_1 outer one in counter-clockwise direction

C_2 inside one in clockwise direction

$\vec{F}$ C^1-v.f. on D.

We claim:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \left(\oint_{C_1} + \oint_{C_2}\right) P\,dx + Q\,dy.$$

Idea: Cut the region (destroy the hole) by adding C_3 and C_4. Then

D_ε becomes simply conn.

$$\iint_{D_\varepsilon} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{C_1(\varepsilon)} + \int_{C_2(\varepsilon)} + \int_{C_3(\varepsilon)} + \int_{C_4(\varepsilon)} (P\,dx + Q\,dy).$$

As the small parameter $\varepsilon \to 0$,

$$D_\varepsilon \to D$$

$$C_1(\varepsilon), C_2(\varepsilon) \to C_1, C_2$$

$$C_3(\varepsilon) \to C_3$$

$$C_4(\varepsilon) \to C_4 = -C_3$$

$$\therefore \int_{C_3(\varepsilon)} + \int_{C_4(\varepsilon)} (\ ) \to \int_{C_3} - \int_{C_3} (\ ) = 0$$

$$\therefore \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \left(\oint_{C_1} + \oint_{C_2}\right) P\,dx + Q\,dy$$

Theorem 4 Let D be a region bdd by $C_1, \ldots, C_n$ where C_1 is outside closed loop and $C_2, \ldots, C_n$ are inside and disjoint. Then

$$\sum_{j=1}^{n} \oint_{C_j} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

when C_1 is anticlockwise direction and $C_2, \ldots, C_n$ is clockwise direction.